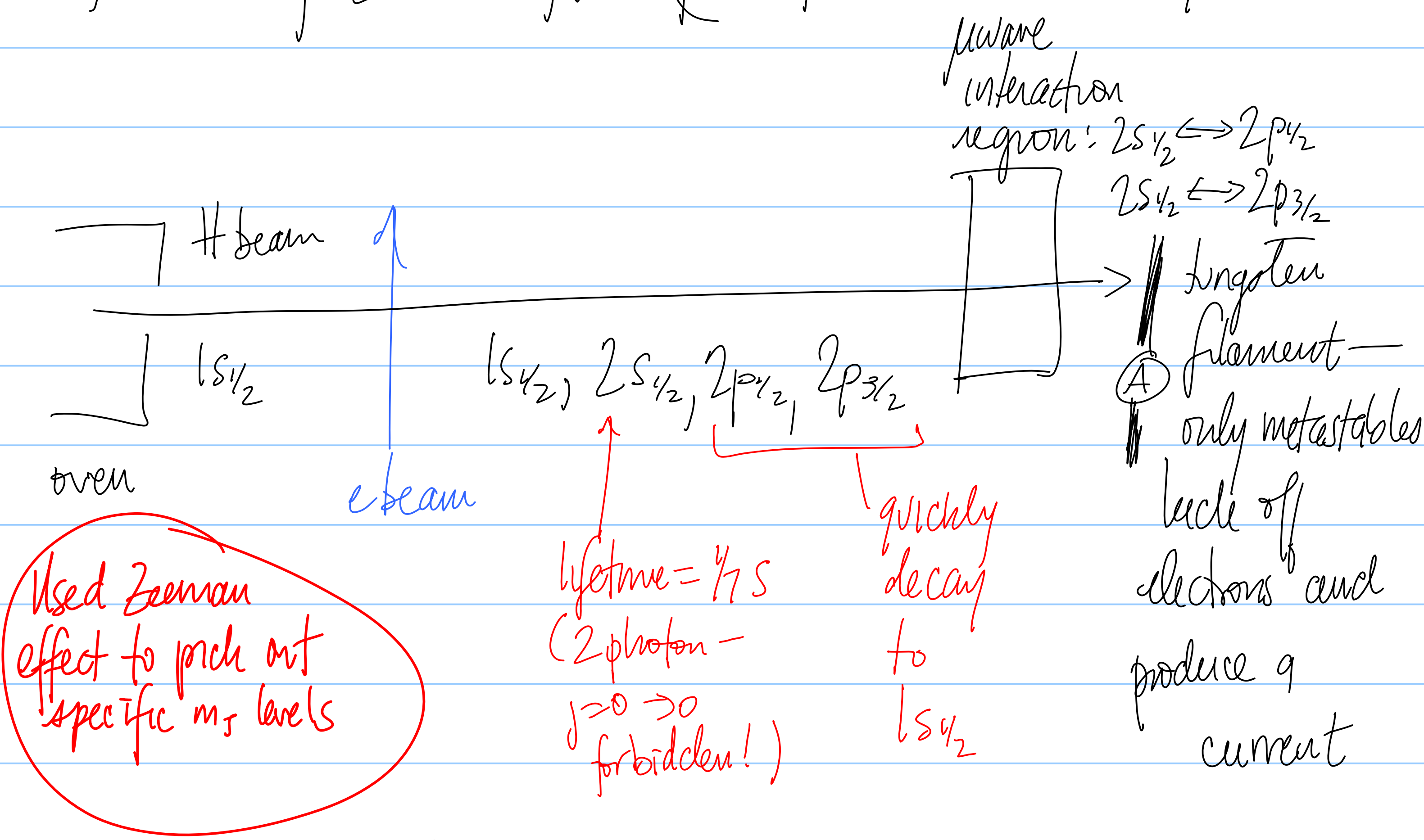
# Lamb shift



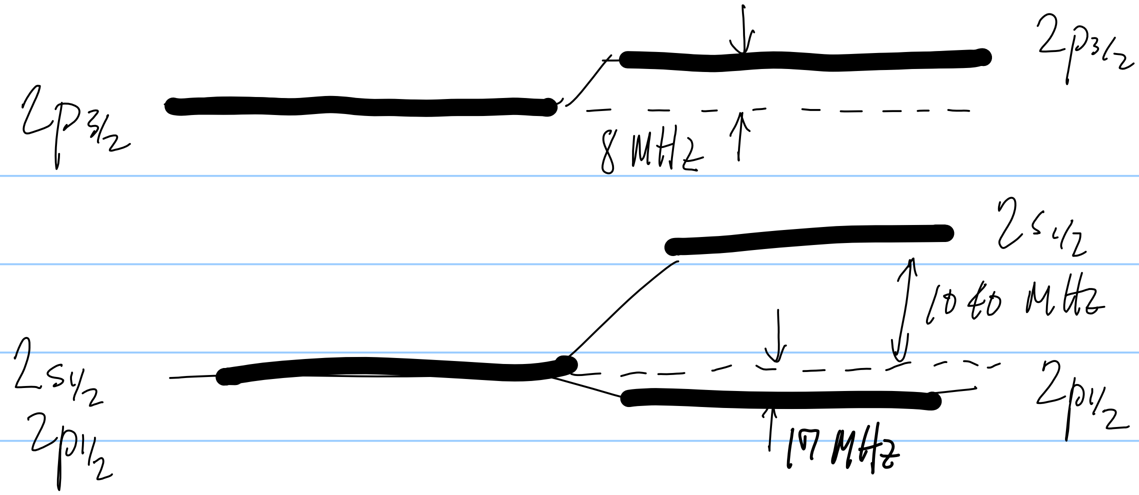
Need QED to explain this effect — in fact, Lamb's measurement in 1947 drove the development of QED!

→ think of Lamb shift = (part of) QED − Dirac equation




Lamb and Retherford measured $|E_{2p_{1/2}} - E_{2s_{1/2}}| \sim 1\ \text{GHz}$,
Lamb later (1953) found $\Delta E = 1057.77(10)$ MHz

Many other measurements — Hansch & Schawlow ('72), ....

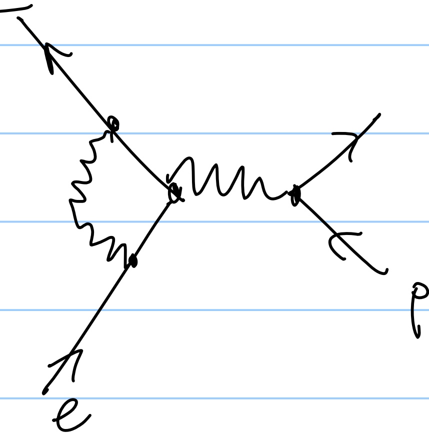



First treatment by Bethe, then later development by Schwinger, Weisskopf, and Feynman into QED.

Simple (incorrect model) from Welton:

in cgs

In QED, main contribution:




In this model, the zero-point energy of the electro-magnetic vacuum modes smear out the electron in space

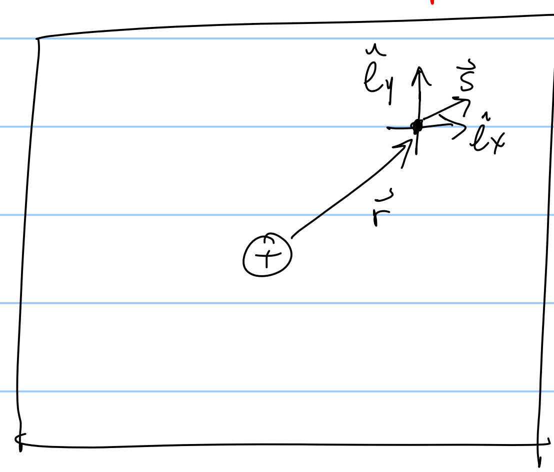
$$\rho(\nu)d\nu = 8\pi \frac{\nu^2}{c^3} d\nu$$ density of states for photon modes

$$W_\nu = \tfrac{1}{2} h\nu\rho(\nu) = 4\pi \frac{h\nu^3}{c^3}$$ zero-point energy density

$$W_\nu = \frac{1}{8\pi}\left(\overline{E^2} + \overline{B^2}\right) = \frac{1}{8\pi} E_\nu^2$$

peak electric field

so, $$E_\nu^2 = \frac{32\pi^2 h\nu^3}{c^3}$$




look at small perturbations to the electron's motion:

in one direction: $$m\ddot{s}_\nu = eE_\nu \cos 2\pi\nu t$$

Fourier decomposition into motion at different frequencies $\left(\vec{s} = \sum_{i=x,y,z,\nu} s_{\nu,i}\,\hat{e}_i\right)$

$\downarrow$ integrate

$$s_\nu = -\frac{eE_\nu}{m}\frac{1}{(2\pi\nu)^2}\cos(2\pi\nu t)$$

$$\langle s_\nu^2\rangle = \frac{1}{2}\left[\frac{eE_\nu}{m(2\pi\nu)^2}\right]^2 = \frac{e^2 E_\nu^2}{32\pi^4 m^2 \nu^4} = \frac{e^2 h}{\pi^2 m^2 c^3}\frac{1}{\nu}$$

This causes a change in the average Coulomb potential experienced by the electron:

$$\Delta V = \overline{V(\vec{r}+\vec{s}_\nu)} - V(r)$$

i, j refer to cartesian directions

$$V(\vec{r}+\vec{s}) = V(\vec{r}) + \underbrace{\vec{\nabla} V \cdot \vec{s}_\nu}_{\text{averages away}} + \underbrace{\tfrac{1}{2}\sum_{ij} \frac{\partial^2 V}{\partial s_{\nu,i}\,\partial s_{\nu,j}}\, s_{\nu,i}\, s_{\nu,j}}_{\text{only } \frac{\partial^2 V}{\partial s_i^2} s_i^2 \text{ terms not zero}} + \dots$$

$\nu$ is the frequency

Taylor expansion

(motion is random & isotropic)

$$\overline{V(\vec{r}+\vec{s})} \simeq V(\vec{r}) + \tfrac{1}{2}\sum \frac{\partial^2 V}{\partial s_{\nu,i}^2}\, \overline{s_{\nu,i}^2}$$

for isotropic oscillator, $\overline{s_{\nu,i}^2} = \tfrac{1}{3}\overline{s_\nu^2}$

with $s_\nu = \sqrt{s_{\nu,x}^2 + s_{\nu,y}^2 + s_{\nu,z}^2}$

$$\overline{V(\vec{r}+\vec{s})} = \frac{\overline{s_\nu^2}}{6}\nabla^2 V(r) + V(r)$$

for a hydrogenic atom: $\nabla^2 V = 4\pi Z e\,\delta(\vec{r})$

$$\Delta E_\nu = e \cdot 4\pi Z e\, \frac{\overline{s_\nu^2}}{6}\, \underbrace{\langle nlm|\delta(\vec{r})|nlm\rangle}_{\text{only non-zero for } l=0 \text{ (s-states)}}$$

(p-states do have shift, but it's much smaller!)

recall: $|\psi_{n,0,0}(0)|^2 = \dfrac{Z^3}{\pi n^3 a_0^3}$

$$\Delta E_\nu = 4\pi Z e^2 \frac{\overline{S_\nu^2}}{Ce} \frac{Z^3}{\pi n^3 a_0^3}$$

$$= \frac{2}{3} \frac{Z^4 e^2}{n^3 a_0^3} \frac{e^2 h}{\pi^2 m^2 c^3} \frac{1}{\nu} = \frac{2}{3} \frac{Z^4 e^4 h}{n^3 a_0^3 \pi^2 m^2 c^3} \frac{1}{\nu}$$

$$\int_{\nu_{min}}^{\nu_{max}} \Delta E_\nu \, d\nu = \frac{2}{3} \frac{e^4}{m^2 c^3} \frac{Z^4}{\pi^2} \frac{h}{n^3 a_0^3} \ln\left(\frac{\nu_{max}}{\nu_{min}}\right)$$

What do we for the frequency cutoffs?

low end - frequency associated with electron orbit - this is the lowest frequency around, so it's a guess

high end - frequency associated with rest mass energy - just a guess (it's the highest frequency around)

plug & chug - $\Delta E = 1600$ MHz for 2s state in H - not bad!

Still working on these measurements!

now at kHz level (Hänsch)

- look at what happens if proton has structure
- can see higher order diagrams